PRIME IDEALS OF ORE EXTENSIONS

ANDRE LEROY
UNIVERSITY OF VALENCIENNES
LE MONT HOUY
59326 VALENCIENNES
FRANCE.

JERZY MATCZUK
UNIVERSITY OF WARSAW
00-901 WARSAW, PKIN
POLAND.

Dedicaded to the memory of Professor Robert B. Warfield.

Abstract. For the Ore extension $R[t, S, D]$, where R is a prime ring, we describe prime ideals having zero intersection with R.

Introduction. The structure of prime ideals of various kinds of ring extensions has been investigated during the last few years. Normalizing extensions ([11]), crossed products ([2],[10]), enveloping rings ([11],[12]) and Ore extensions ([1],[3],[4],[5],[6],[7]) were, in particular, studied.

In [7], [8] primes of Ore extensions over commutative noetherian rings were considered. In [2], [5] and [12], prime ideals, disjoint from the coefficient ring, of Ore extensions of derivation type were described. The case of Ore extensions of automorphism type has been dealt recently in [1], [4]. The aim of this paper is to study these prime ideals of Ore extensions, which have zero intersection with coefficient ring. The methods we use are based on [9].

Throughout the paper R will denote a prime ring while T will stand for the symmetric quotient ring of R. Recall that the left Martindale quotient ring of R is defined as $Q(R) = \varinjlim_{I\in\mathcal{F}} Hom_R({}_RI, {}_RR)$, where $\mathcal{F}$ is the filter of all non–zero ideals of R, and T can be considered as a subring of $Q(R)$ consisting of such elements $q \in Q(R)$ that $qI \subset R$ for some ideal $I \in \mathcal{F}$ depending on q.

The paper was written while the authors were staying at Institut d'Estudies Catalans, Centre de Recerca Matematica, Spain.

S and D will stand for an automorphism and S–derivation of R, respectively. Recall that an S–derivation D is an endomorphism of the additive group of R such that

$$D(ab) = D(a)b + S(a)D(b) \text{ for all } a, b \in R \quad .$$

In case S is the identity, D is an ordinary derivation. For each $c \in R$ we will denote by $D_{c,S}$ the S–derivation of R defined by $D_{c,S}(a) = ca - S(a)c$ for all $a \in R$. The Ore extension $R[t, S, D]$ is the ring of polynomials in t over R, with multiplication determined by the rule

$$ta = S(a)t + D(a) \text{ for all } a \text{ in } R \quad .$$

For $f(t) \in R[t, S, D]$, $deg\, f(t)$ will denote the degree of the polynomial $f(t)$.

It is well–known that both S and D have unique extensions to T. Therefore we can consider the over ring $T[t, S, D]$ of $R[t, S, D]$. We will give a complete description of T–disjoint prime ideals of $T[t, S, D]$. Next we will present a one–to–one correspondence between T–disjoint primes of $T[t, S, D]$ and R–disjoint primes of $R[t, S, D]$, provided one of the following conditions is satisfied :

a) R is symmetrically closed (i.e. $T = R$)
b) R is left and right nœtherian
c) R satisfies the descending chain condition on *two*–sided ideals
d) S and D commute and another minor technical assumption (cf. Prop. 2.9.)

The above results lead to a full description of R–disjoint prime ideals of $R[t, S, D]$ in these cases.

Invariant polynomials. Recall that if $f(t) \in R[t, S, D]$ is a monic polynomial, then $f(t)$ is invariant if

(i) $f(t)t = (t + \alpha)f(t)$ for some $\alpha \in R$.

and

(ii) for any $r \in R \quad f(t)r = S^n(r)f(t)$, where $n = deg\, f(t)$.

If $f(t) \in R[t, S, D]$ is monic invariant then clearly the left ideal generated by $f(t)$ is two–sided and moreover $R[t, S, D]f(t) = f(t)R[t, S, D]$. Conversely, as the following lemma shows, it is possible to associate in a unique way a monic invariant polynomial to any non–zero ideal of $R[t, S, D]$.

Lemma 1.1. *(Prop. 2.1, Cor. 2.2 [9])*
(1) For any non–zero ideal I of $R[t, S, D]$ there exists a unique monic invariant polynomial $f_I(t) \in T[t, S, D]$ having the following properties:

(i) *$deg\, f_I(t) = min\{deg\, g(t) | 0 \neq g(t) \in I\} = n$ and every polynomial $g(t) \in I$ of degree n is of the form $g(t) = af_I(t)$ for some $a \in R$.*
(ii) *$I \subset T[t, S, D]f_I(t) \cap R[t, S, D]$.*

(2) If I is an ideal of $T[t, S, D]$ then the polynomial $f_I(t)$ defined in (1) belongs to $T[t, S, D]$. ■

The polynomial $f_I(t)$ from the above lemma will be called the invariant polynomial associated to I.

In the sequel we will need the following simple observation.

Lemma 1.2. *Let $I \subset J$ be non-zero ideals of $R[t, S, D]$. Then there is a monic invariant polynomial $h(t) \in T[t, S, D]$ such that $f_I(t) = h(t)f_J(t)$.*

Proof. By Lemma 1.1, there is a non-zero ideal A of R such that $Af_I(t) \subset I$. Since $I \subset J$, Lemma 1.1 yields that for any $a \in A$ there is $g_a(t) \in T[t, S, D]$ such that

$$a\, f_I(t) = g_a(t)\, f_J(t) \quad .$$

$f_J(t)$ is a monic polynomial, thus we can divide $f_I(t)$ on the right by $f_J(t)$ getting $f_I(t) = h(t)f_J(t) + r(t)$ for some $h(t), r(t) \in T[t, S, D]$ with $deg\, r(t) < deg\, f_J(t)$.

Therefore, for any $a \in A$ we have

$$g_a(t)f_J(t) = af_I(t) = ah(t)f_J(t) + ar(t)$$

and, consequently, $(g_a(t) - ah(t))f_J(t) = ar(t)$. Comparing degrees of polynomials appearing on both sides in the above equality we get $ar(t) = 0$ for all $a \in A$. This implies $r(t) = 0$ and $f_I(t) = h(t)f_J(t)$. Now one can easily check that $h(t)$ is a monic invariant polynomial. ■

Let $M(t) \in T[t, S, D]$ denote a monic invariant polynomial of minimal non-zero degree, provided such polynomial exists; otherwise $M(t) = 1$.

In order to describe prime ideals in $R[t, S, D]$ we will need a description of the center Z of $T[t, S, D]$ and some properties of invariant polynomials. $C_{S,D}$ will denote the ring of all central elements in T which are S and D invariant.

Proposition 1.3. *(1)* (Th. 3.6 and 3.7 [9]) *There exist an invertible $\lambda \in T$ and $\ell \geq 0$ such that $Z = C_{S,D}[z]$, where $z = \lambda M(t)^{\ell}$. Moreover, $Z \neq C_{S,D}$ iff $M(t) \neq 1$ and a non-zero power of S is an inner automorphism of T.*

(2) (Prop. 3.4 [9]) *Every monic invariant polynomial $f(t) \in T[t, S, D]$ can be written in the form $f(t) = \alpha\omega(z)M(t)^m$, where $\alpha \in T$ is invertible, $m \geq 0$ and $\omega(z)$ is a monic polynomial in the center $C_{S,D}[z]$ of $T[t, S, D]$.* ■

We will say that the center Z of $T[t, S, D]$ is non-trivial if $Z \neq C_{S,D}$.

In the following lemma the notation will be as in the above proposition. Additionally we will assume that Z is non-trivial.

Lemma 1.4. *Let $f(t) \in T[t,S,D]$ be a monic non–constant invariant polynomial and $Z = C_{S,D}[z]$ denote the center of $T[t,S,D]$. The following conditions are equivalent:*

(i) *$f(t)$ can not be presented as a product of two monic non–constant invariant polynomials.*

(ii) *Either $f(t) = M(t)$ or there is an invertible $\beta \in T$ such that $\beta f(t) \in C_{S,D}[z]$ is a monic irreducible polynomial in $C_{S,D}[z]$ different from z.*

Proof. $(i) \to (ii)$. By Proposition 1.3, $f(t) = \alpha\omega(z)M(t)^m$ for some invertible $\alpha \in T$, a monic polynomial $\omega(z) \in C_{S,D}[z]$ and $m \geq 0$. Since both $f(t)$ and $M(t)$ are monic polynomials, $\alpha\omega(z)$ is a monic polynomial in $T[t,S,D]$ and, clearly, $\alpha\omega(z)$ is an invariant polynomial. Therefore, the assumption on $f(t)$ yields that either $f(t) = M(t)$ or $f(t) = \alpha\omega(z)$. Suppose that $f(t) = \alpha\omega(z)$, i.e. $\alpha^{-1}f(t) = \omega(z) \in C_{S,D}[z]$. First we will show that $\alpha^{-1}f(t)$ is irreducible as a polynomial in $C_{S,D}[z]$. Assume that $\alpha^{-1}f(t) = \omega_1(z)\omega_2(z)$ for some $\omega_i(z) \in C_{S,D}[z]$ with $deg_z\omega_i(z) > 0$, $i = 1,2$. We will treat $\omega_i(z)$'s as polynomials in t and denote $\bar{\omega}_i(t) = \omega_i(z)$, $i = 1,2$. By Proposition 1.3, the leading coefficients α_1, α_2 of polynomials $\bar{\omega}_1(t)$, $\bar{\omega}_2(t)$ are invertible in T and $\alpha_1^{-1}\bar{\omega}_1(t)$, $\alpha_2^{-1}\bar{\omega}_2(t)$ are monic invariant polynomials. Therefore we can present $f(t)$ in the form

$$f(t) = \alpha\bar{\omega}_1(t)\bar{\omega}_2(t) = (\alpha\alpha_2\alpha_1)(\alpha_1^{-1}\bar{\omega}_1(t))(\alpha_2^{-1}\bar{\omega}_2(t)) \quad .$$

Since $f(t)$ is monic, $\alpha\alpha_2\alpha_1 = 1$ and the above equality shows that $f(t)$ can be decomposed into a product of two non–constant monic invariant polynomials. This contradicts our assumption and establishes $\alpha^{-1}f(t)$ is an indecomposable polynomial in $C_{S,D}[z]$.

Recall that $z = \lambda M(t)^\ell$ for some invertible $\lambda \in T$ and $\ell > 0$. Using this presentation of z and the assumption on $f(t)$ it is easy to check that if $\beta f(t) = z$ for some invertible $\beta \in T$ then $\beta = \lambda$ and $\ell = 1$, i.e. $f(t) = M(t)$. This completes the proof of the implication $(i) \to (ii)$.

$(ii) \to (i)$. If $f(t) = M(t)$ then clearly $f(t)$ can not be decomposed into the product of two non–constant monic invariant polynomials. Suppose that there is an invertible $\beta \in T$ such that $\beta f(t) \in C_{S,D}[z]$ is a monic irreducible polynomial in z different from z.

Let $1 \neq f_1(t), f_2(t) \in T[t,S,D]$ be monic invariant polynomials such that $f(t) = f_1(t)f_2(t)$. We will show that $f_2(t) = 1$. By making use of Proposition 1.3 we can write polynomials $f_1(t), f_2(t)$ in the form $f_i(t) = \alpha_i\omega_i(z)M(t)^{m_i}$ where $\alpha_i \in T$ is invertible, $\omega_i(z)$ is a monic polynomial in $C_{S,D}[z]$, $m_i \geq 0$; $i = 1,2$. Then

$$\beta f(t) = \beta\alpha_1\omega_1(z)M(t)^{m_1}\alpha_2\omega_2(z)M(t)^{m_2} = \omega_1(z)\omega_2(z)\gamma M(t)^{m_1+m_2}$$

for some invertible $\gamma \in T$. Since $\beta f(t)$, $\omega_1(z)$, $\omega_2(z)$ are monic in z, central polynomials, $\gamma M(t)^{m_1+m_2}$ is a monic central polynomial in z. Using the description of central polynomials it is easy to see that $\gamma M(t)^{m_1+m_2} = z^k$ for some $k \geq 0$. Therefore $\beta f(t) = \omega_1(z)\omega_2(z)z^k$. Because $\beta f(t)$ is an irreducible polynomial in $C_{S,D}[z]$, we get:

(*) $\quad k = 0$, since otherwise $\omega_1(z) = \omega_2(z) = 1$ and $\beta f(t) = z$.

(**) $\quad \omega_1(z) = 1$ or $\omega_2(z) = 1$.

Since $k = 0$, $f_i(t) = \alpha_i \omega_i(z)$, $i = 1, 2$. Now the condition (**) together with the fact that $f_1(t) \neq 1$ forces $\omega_2(z) = 1$. It means that $f_2(t) = 1$ and establishes the lemma. ■

Up to the end of this section we will additionally assume that S and D commute. It is well known that in this case S can be extended to an automorphism of $T[t, S, D]$ by setting $S(t) = t$ and D can be extended to an S-derivation of $T[t, S, D]$ by $D(t) = 0$.

In the next lemma we will describe the set of all monic invariant polynomials of minimal non-zero degree and study the additive commutator $[M(t), t]$.

Lemma 1.5. *Suppose that $M(t), M'(t) \in T[t, S, D]$ are monic invariant polynomials of minimal non-zero degree. Then :*

(i) *$M'(t) = M(t) + c$ for some $c \in T$. If $c \neq 0$ then c is invertible in T and S^n is an inner automorphism of T determined by c, where $n = deg\, M(t)$.*

(ii) *$M(t) = t + b$ if and only if $D = D_{-b;S}$ (i.e. $D(x) = S(x)b - bx$ for all $x \in T$).*
If moreover $S(M(t)) \neq M(t)$ then S is an inner automorphism of T and $T[t, S, D] \cong T[t']$

(iii) *$M(t)t = tM(t)$ if either $deg\, M(t) > 1$ or $S(M(t)) = M(t)$*

(iv) *If $M(t)t = tM(t)$ then there exists $c \in T$ such that $S(c) = c$, $D(c) = 0$ and $S(M(t)) = M(t) + c$, $D(M(t)) = -ct$.*

Proof. (i) By Proposition 1.3 $M'(t) = \alpha\omega(z)M(t)^m$ for some $m \geq 0$ where $\alpha \in T$ is invertible and $\omega(z) \in C_{S,D}[z]$. Comparing degrees of polynomials appearing in the above equality we obtain that either $m = 1$ and $M'(t) = M(t)$ or $m = 0$ and $M'(t) = \alpha\omega(z)$. In the second case, by using again Proposition 1.3, we get $M'(t) = M(t) + c$ for some $c \in T$.

Now, for any $r \in R$,

$$cr = (M'(t) - M(t))r = S^n(r)(M'(t) - M(t)) = S^n(r)c \quad ,$$

where $n = deg\, M(t)$. It means that the element c normalizes R. Now the statement (i) follows from the fact that non–zero R–normalizing elements from T are invertible.

Notice that in the proof of (i) we have not used the assumption that S commutes with D.

(ii) Since $M(t) = t + b$ is invariant we have, for any $x \in T, M(t)x = S(x)M(t)$ and a comparison of independant terms on both sides of this equation leads to $D(x) + bx = S(x)b$ i.e. $D = D_{-b;S}$.
Conversely if $D = D_{-b;S}$ then we easily verify that $(t + b)x = S(x)(t + b)$ for any $x \in T$ and that $(t + b)t = (t + c)(t + b)$ where $c = b - S(b)$. This shows that $M(t) = t + b$ is invariant.
Now, $S(M(t)) = S(t + b) = t + S(b) = t + b + S(b) - b = M(t) - c$. Hence $S(M(t)) \neq M(t)$ if and only if $c \neq 0$. Since $S(M(t))$ is obviously an invariant polynomial of minimal non zero degree, part (i) above shows that if $S(M(t)) \neq M(t)$ then S is an inner automorphism of T induced by c and one can check that $c^{-1}(t + b)$ is a central polynomial in $T[t, S, D]$. This yields $T[t, S, D] \cong T[t']$ for $t' = c^{-1}(t + b)$.

(iii) &(iv). By (i), $S(M(t)) = M(t) + c$ for some $c \in T$. Let $a \in T$ be such that $M(t)t = (t + a)M(t)$ ($M(t)$ is monic invariant). Then

$$\begin{aligned} M(t)t &= t\, M(t) + a\, M(t) = S(M(t))t + D(M(t)) + a\, M(t) = \\ &= (M(t) + c)t + a\, M(t) + D(M(t)) \end{aligned}$$

Hence

$$D(M(t)) = -a\, M(t) - ct \tag{1}$$

If $deg\, M(t) > 1$, then since $deg\, D(M(t)) < deg\, M(t)$ the equation (1) shows that $a = 0$ i.e. $M(t)t = tM(t)$.
If $deg\, M(t) = 1$ but $S(M(t)) = M(t)$ then $c = 0$ and (1) implies that $a = 0$. This completes part (iii).
Now if $M(t)t = tM(t)$ the element a defined above is equal to zero and (1) shows that $D(M(t)) = -ct$. Where $c \in T$ is such that $S(M(t)) = M(t) + c$. If $c \neq 0$ part (i) of this lemma implies that $S(c) = c$ and, by comparing $S(D(M(t)))$ and $D(S(M(t)))$, we get $D(c) = 0$.

Example 1.6. Let us give an example of a monic invariant polynomial of minimal degree $M(t)$ in an Ore extension $T[t, S, D]$ such that $S \circ D = D \circ S$ but $M(t)t \neq tM(t)$. Lemma 1.5 shows that the degree of such a polynomial must be one. Consider the polynomial ring $k[c]$ over a field k and let $\delta = c^2 \frac{d}{dc}$. Define an automorphism over $R = k[c][b, \delta]$ by putting $S(p(c)) = p(c)$ for $p(c) \in k[c]$ and $S(c) = b - c$. It is easy to observe that S is a well defined automorphism of R and that $S \circ D_{-b;S} = D_{-b;S} \circ S$. Let T be the symmetric Martindale ring of quotients (e.g. $T = R$ if char $k = 0$ since in this case R is simple). In the Ore extension $T[t, S, D_{-b;S}]$ the polynomial $t + b$ is invariant. Since $D_{-b;S}(b) = -cb$ we get $tb = (b - c)t - cb$ and so

$$(t + b)t = t^2 + tb + ct + cb = (t + c)(t + b)$$

This shows that $(t + b)t \neq t(t + b)$. ■

In the sequel we will use the following simple technical observation.

Lemma 1.7. *Suppose that $\lambda \in T$ and $g(t) \in T[t, S, D]$ is not zero divisor. If both $g(t)$ and $\lambda g(t)$ commute with t, then $S(\lambda) = \lambda$ and $D(\lambda) = 0$.*

Proof. Suppose that $g(t)$ and $\lambda g(t)$ commute with t. Using regularity of $g(t)$ it is standard to see that λ commutes with t. This implies the thesis. ■

The following lemma is of independent interest.

Lemma 1.8. *Suppose that either deg $M(t) > 1$ or $S(M(t)) = M(t)$ and let $g(t)$ be a monic invariant polynomial. Then $g(t)$ commutes with t.*

Proof. If the center Z of $T[t, S, D]$ is trivial then, by Proposition 1.3, every monic invariant polynomial is a power of $M(t)$. In this case the thesis is a consequence of Lemma 1.5.

Suppose Z is non–trivial, i.e. $Z = C_{S,D}[z]$, where $z = \lambda M(t)^\ell$ for some invertible $\lambda \in T$, $\ell > 0$. By making use of Lemmas 1.5 and 1.7 we get

$$S(\lambda) = \lambda \text{ and } D(\lambda) = 0 \qquad (*)$$

Let $g(t)$ be a monic invariant polynomial. Then, by Proposition 1.3, $g(t) = \alpha\omega(z)M(t)^m$ for some invertible $\alpha \in T$, $\omega(z) \in C_{S,D}[z]$, $m \geq 0$. Since $M(t)$ is invariant. Property (*) shows that $M(t)\lambda = \lambda M(t)$, hence if we write $\omega(z) = \sum_{i=0}^{s} \alpha_i z^i$, $\alpha_i \in C_{S,D}$ we then get

$$g(t) = \alpha\omega(z)M(t)^m = \alpha \sum_{i=0}^{s} \alpha_i \lambda^i M(t)^{i+m}$$

A comparison of leading coefficients shows that $1 = \alpha\alpha_s\lambda^s$ and so α is S and D invariant. Hence t commutes with α, $\omega(z)$ and,thanks to Lemma I.5 (iii), also with $M(t)$. This proves that t commutes with $g(t)$. ■

Let us recall that an ideal I of R is called S, D stable if $S(I) = I$ and $D(I) \subseteq I$.

Lemma 1.9. *Suppose that either $deg\, M(t) > 1$ or $S(M(t)) = M(t)$. Let $g(t) = \sum_{i=0}^{s} q_i t^i \in T[t, S, D]$ be either a monic invariant polynomial or $g(t) \in Z$ - the center of $T[t, S, D]$. Then there is a non-zero S, D stable ideal I of R such that Iq_i, $q_i I \subset R$ for $0 \leq i \leq s$.*

Proof. Assume that $0 \neq J$ is an S, D–stable ideal of R such that $Jg(t) \subset R[t, S, D]$. Since $S(J) = J$, $g(t)J = Jg(t) \subset R[t, S, D]$. Now it is straightforward to verify (cf. Lemma 1.3 [9]) that $q_i J \subset R$ for $0 \leq i \leq s$. Therefore in order to establish the lemma it is enough to find a non–zero S, D–stable ideal I of R such that $Ig(t) \subset R[t, S, D]$. First we will find such an ideal for $g(t) = M(t)$. By Lemma 1.5, $S(M(t)) = M(t) + c$ where the element $c \in T$ satisfies: $S(c) = c$, $D(c) = 0$ and $cR = Rc$. Define $I = \{r \in R | r\, M(t) \in R[t, S, D]$ and $rc \in R\}$. Clearly I is a left ideal of R and $I \neq 0$ by definition of T. Since both $M(t)$ and c normalize R, I is an ideal of R.

Let $r \in I$. Since $S(c) = c$, $S(r)c = S(rc) \in R$ and $S(r)M(t) = S(rM(t)) - S(r)c \in R[t, S, D]$. This shows that $S(I) \subset I$. Applying the same argument to S^{-1} we obtain $S(I) = I$. By Lemma 1.5 $D(M(t)) = -ct$. Hence $D(r)M(t) = D(r\, M(t)) + S(r)ct \in R[t, S, D]$. Since $D(c) = 0$, $D(r)c \in R$. This completes the proof that

$$I \text{ is an } S, D\text{–stable ideal of } R \text{ such that } IM(t) \subset R[t, S, D]\,. \qquad (*)$$

If the center Z of $T[t, S, D]$ is equal to $C_{S,D}$ then, by Proposition 1.3, every monic invariant polynomial is a power of $M(t)$. Thus the statement $(*)$ yields the thesis in this case.

Suppose that $Z \neq C_{S,D}$. Then, by Proposition 1.3, $Z = C_{S,D}[z]$ where $z = \lambda M(t)^\ell$ for some $\ell > 0$ and an invertible element $\lambda \in T$. By Lemma 1.5, $M(t)^\ell$ commutes with t. Hence Lemma 1.7 shows that $S(\lambda) = \lambda$ and $D(\lambda) = 0$. Now using the above property of λ together with the statement $(*)$ it is easy to complete the proof for $g(t) \in C_{S,D}[z]$.

Finally let $g(t)$ be a monic invariant polynomial. Then, by Proposition 1.3, $g(t) = \alpha\omega(z)M(t)^m$ for some $m \geq 0$ where $\alpha \in T$ is invertible and $\omega(z) \in C_{S,D}[z]$. Lemma 1.8 implies that $g(t)$ commutes with t. Since the polynomial $\omega(z)M(t)^m$ also commutes with t Lemma 1.7 gives us $S(\alpha) = \alpha$ and $D(\alpha) = 0$. Using this property and what has been proved above for $M(t)$ and for central polynomials, it is easy to show that there is a non–zero S, D–stable ideal I of R such that $Ig(t) \subset R[t, S, D]$, as required. ■

Prime ideals of R[t,S,D]. In this part we give a description of prime ideals of $R[t, S, D]$ having zero intersection with the coefficient ring R. Using Lemma 1.1 it is standard to prove the following:

Proposition 2.1. *For the ring $R[t,S,D]$ the following conditions are equivalent:*

(i) 0 *is the only R–disjoint prime ideal of $R[t,S,D]$.*
(ii) *$R[t,S,D]$ has no non–zero R–disjoint ideals.*
(iii) *$T[t,S,D]$ does not contain non–constant monic invariant polynomial.* ■

The equivalence given in the above proposition can be also expressed in terms of properties of S and D (cf. Th. 2.6 [9]).

Because of Proposition 2.1 we will further assume that $R[t,S,D]$ has non–zero R–disjoint ideals. Notice that in this case there are non–constant monic invariant polynomials in $T[t,S,D]$, so $M(t) \neq 1$.

$Spec_\circ(R[t,S,D])$ will denote the set of all prime ideals of $R[t,S,D]$ which are R–disjoint.

$Max_\circ(R[t,S,D])$ will stand for the set of all maximal ideals among R–disjoint ideals.

Since R is prime, it is easy to check that

$$Max_\circ(R[t,S,D]) \subseteq Spec_\circ(R[t,S,D]).$$

This observation will be used freely in the sequel.

We will continue to use the notation from Proposition 1.3. In particular the center Z of $T[t,S,D]$ is non–trivial if it is not contained in T. In this case $Z = C_{S,D}[z]$ where $z = \lambda M(t)^\ell$ for some invertible $\lambda \in T$ and $\ell > 0$.

Theorem 2.2. *For a non–zero ideal P of $T[t,S,D]$ the following conditions are equivalent:*

(i) $P \in Spec_\circ(T[t,S,D])$
(ii) $P \in Max_\circ(T[t,S,D])$
(iii) $P = f(t)T[t,S,D]$ *where $f(t) \in T[t,S,D]$ is either equal to $M(t)$ or the center Z of $T[t,S,D]$ is non–trivial and there is an invertible $\beta \in T$ such that $\beta f(t) \in Z = C_{S,D}[z]$ is a monic irreducible polynomial (as a polynomial in z) different from z.*

Proof. $(i) \to (iii)$. Let $0 \neq P \in Spec_\circ(T[t,S,D])$. By Lemma 1.1,

$$P \subset f_P(t)T[t,S,D] \qquad (*)$$

where $f_P(t)$ denotes the monic invariant polynomial associated to P.

We will show that $P = f_P(t)T[t,S,D]$. Define

$$P_\circ = \{h(t) \in T[t,S,D] | f_P(t)h(t) \in P\}\ .$$

Since $f_P(t)$ is invariant, $P_\circ$ is an ideal of $T[t,S,D]$ and, by Lemma 1.1, $P_\circ \cap T \neq 0$. Clearly we have $(f_P(t)T[t,S,D])P_\circ \subset f_P(t)P_\circ \subset P$ and $P_\circ \not\subseteq P$. Hence primeness of P and $(*)$ establish $f_P(t)T[t,S,D] = P$. The fact that non–constant monic invariant polynomials generate two–sided ideals and primeness of P implies that $f_P(t)$ can not be decomposed into the product of two non–constant monic invariant polynomials. Now Lemma 1.4 completes the proof of $(i) \to (iii)$.

$(iii) \to (ii)$. Suppose that $P = f(t)T[t,S,D]$, where $f(t)$ is described as in (iii). If $\beta f(t)$ is a central polynomial then the leading coefficient α of $\beta f(t)$ normalizes R, so α is invertible in T and $\alpha^{-1}\beta f(t)$ is a monic invariant polynomial. Using Lemma 1.1, it is easy to see that $\alpha^{-1}\beta f(t) = f_P(t)$. Thus, by Lemma 1.4, $f_P(t)$ can not be decomposed into a product of two non–constant monic invariant polynomials.

Now let I be a non–zero T–disjoint ideal of $T[t,S,D]$ such that $P \subset I$. Then, by Lemmas 1.2 and 1.1, $f_P(t) = h(t)f_I(t)$ for some monic invariant polynomial $h(t) \in T[t,S,D]$. This implies that $f_P(t) = f_I(t)$, since $f_I(t) \neq 1$ and f_P is indecomposable. Using again Lemma 1.1 we have $P \subset I \subset f_I(t)T[t,S,D] = f_P(t)T[t,S,D] = P$. Thus $P = I$ and $P \in Max_\circ(T[t,S,D])$.

$(ii) \to (i)$. This implication is a direct consequence of primeness of T and of the fact that for $P \in Max_\circ(T[t,S,D])$ every ideal strictly containing P has a non–zero intersection with T. ■

Combining Theorem 2.2 and Propositions 2.1 and 1.3 we get the following:

Corollary 2.3. *Let $Spec(Z)$ denote the set of all prime ideals of Z – the center of $T[t,S,D]$. There is one–to–one correspondence between $Spec_\circ(T[t,S,D])$ and $Spec(Z)$ except the case when $T[t,S,D]$ has non–zero T–disjoint ideals and no non–zero power of S is an inner automorphism of T. In this case $Spec(Z) = \{0\}$ but $Spec_\circ(T[t,S,D]) = \{0, M(t)T[t,S,D]\}$.* ■

As a consequence of Theorem 2.2 and Lemma 1.4 we also obtain the following:

Corollary 2.4. *Let $0 \neq P \in Spec_\circ(T[t,S,D])$ then $P = f_P(t)T[t,S,D]$, where $f_P(t)$ is the monic invariant polynomial associated to P.* ■

Now we will pass to the description of $Spec_\circ(R[t,S,D])$. For this some preparation is needed. For an ideal I of $R[t,S,D]$ we define the closure $[I]$ of I as $f_I(t)T[t,S,D] \cap R[t,S,D]$ if $I \neq 0$; otherwise $[I] = 0$. We will say that I is closed if $I = [I]$. This notion was first introduced in [3] for polynomial rings. Using Lemmas 1.1 and 1.2, it is straightforward to verify that the following holds:

Lemma 2.5. *Let I, J be ideals of $R[t,S,D]$. Then:*

(i) $I \subset [I]$
(ii) *If* $I \subset J$ *then* $[I] \subset [J]$.
(iii) $[I]$ *is closed.*
(iv) *If* $I \in Max_\circ(R[t,S,D])$ *then* I *is closed.* ■

Notice also that every $P \in Spec_\circ(T[t,S,D])$ is a closed ideal of $T[t,S,D]$, since, by Theorem 2.2, every non–zero prime ideal belongs to $Max_\circ(T[t,S,D])$.

Lemma 2.6. *Suppose that every $P \in Spec_\circ(R[t,S,D])$ is closed. Then:*

(i) *If $P \neq 0$ then $P \in Spec_\circ(R[t,S,D])$ if and only if $P \in Max_\circ(R[t,S,D])$.*
(ii) *There is one–to–one correspondence between $Spec_\circ(R[t,S,D])$ and $Spec_\circ(T[t,S,D])$ given by*

$$F : Spec_\circ(R[t,S,D]) \to Spec_\circ(T[t,S,D])$$

and

$$G : Spec_\circ(T[t,S,D]) \to Spec_\circ(R[t,S,D])$$

where for $0 \neq P \in Spec_\circ(R[t,S,D])$, $F(P) = f_P(t)T[t,S,D]$ and for $\bar{P} \in Spec_\circ(T[t,S,D])$ $\quad G(\bar{P}) = \bar{P} \cap R[t,S,D]$.

Proof. (ii) First we will show that the maps F and G are well–defined. Let $0 \neq P \in Spec_\circ(R[t,S,D])$ and $f_1(t)$, $f_2(t) \in T[t,S,D]$ be monic invariant polynomials such that $f_P(t) = f_1(t)\,f_2(t)$. Define $I_i = f_i(t)\,T[t,S,D] \cap R[t,S,D]$, $i = 1,2$. Then clearly $P = [P] \subset I_i$, $i = 1,2$. Moreover, using closeness of P, one can show that $I_1 I_2 \subset P$. Hence, by primeness of P, either $I_1 = P$ or $I_2 = P$. It means that either $f_1(t)$ or $f_2(t)$ is equal to $f_P(t)$. This shows that $f_P(t)$ can not be decomposed into a product of two non–constant monic invariant polynomials. Now Theorem 2.2 together with Lemma 1.4 yield $F(P) = f_P(t)T[t,S,D] \in Spec_\circ(T[t,S,D])$, i.e. F is well–defined.

Now take $0 \neq \bar{P} \in Spec_\circ(T[t,S,D])$. By Theorem 2.2 and Lemma 1.4, $\bar{P} = f_{\bar{P}}(t)T[t,S,D]$ and $f_{\bar{P}}(t)$ can not be presented as a product of two non–constant monic invariant polynomials. Let $I \in Max_\circ(R[t,S,D])$ be such that $G(\bar{P}) \subset I$. Then, by Lemma 1.2, there is a monic invariant polynomial $h(t) \in T[t,S,D]$ such that $f_{G(\bar{P})}(t) = h(t)f_I(t)$. Since $f_{G(\bar{P})}(t) = f_{\bar{P}}(t)$, $h(t) = 1$ and $f_{G(\bar{P})}(t) = f_I(t)$. Now $G(\bar{P}) = I$ follows, because both $G(\bar{P})$ and I are closed ideals. This shows that

$$G(\bar{P}) \in Max_\circ(R[t,S,D]) \quad . \qquad (*)$$

Then clearly $G(\bar{P}) \in Spec_o(R[t,S,D])$, i.e. G is well-defined.

Knowing that F and G are well-defined it is standard to complete the proof of the statement (ii).

(i) The inclusion $Max_o(R[t,S,D]) \subset Spec_o(R[t,S,D])$ is clear.

Let $0 \neq P \in Spec_o(R[t,S,D])$. Then, by (ii), $P = G(\bar{P})$ for some suitable $\bar{P} \in Spec_o(T[t,S,D])$ and $(*)$ yields $P \in Max_o(R[t,S,D])$. ■

The above lemma together with Theorem 2.2 provide a description of $Spec_o(R[t,S,D])$ in the case when every $P \in Spec_o(R[t,S,D])$ is closed.

Notice that every $P \in Spec_o(T[t,S,D])$ is a closed ideal of $T[t,S,D]$ since, by Theorem 2.2, every non-zero prime ideal of $T[t,S,D]$ belongs to $Max_o(T[t,S,D])$ and by Lemma 2.5 (iv) every $P \in Max_o(T[t,S,D])$ is closed. In particular if R is a symmetrically closed prime ring (i.e. if $R = T$) then the conditions of Lemma 2.6 are satisfied. We will now show that every $P \in Spec_o(R[t,S,D])$ is closed if one of the following conditions is fulfilled :

1) R is left and right nœtherian
2) R satisfies the descending chain condition on two sided ideals
3) S and D commute and either $deg\, M(t) > 1$ or $S(M(t)) = M(t)$

Proposition 2.7. *Suppose R is left and right nœtherian. Then every ideal $P \in Spec_o(R[t,S,D])$ is closed.*

Proof. In virtue of Lemma 2.5 (iv), it is enough to show that $Spec_o(R[t;SD]) \subseteq Max_o(R[t,S,D])$.
Let $\mathcal{S}$ be the set of regular elements in R and $Q = R\mathcal{S}^{-1} = \mathcal{S}^{-1}R$ be the classical left and right quotient ring. Since R is prime, Q is a left and right artinian simple ring by goldie's theorem. It is standard to extend both S and D to Q and to prove that $\mathcal{S}$ is both a right and left denominator set in $R[t,S,D]$ such that $\mathcal{S}^{-1}(R[t,S,D]) = R[t,S,D]\mathcal{S}^{-1} = Q[t,S,D]$ (cf. [6] Lemmas 1.3 and 1.4). Since $R[t,S,D]$ is both left and right nœtherian there is a (1,1) correspondence between the sets $\{P \in Spec(R[t,S,D]) | P \cap \mathcal{S} = \varnothing\}$ and $Spec(Q[t,S,D])$ (cf.[11] Proposition 2.1.16 (vii)). Since Q is simple and hence also symmetrically closed we have $Spec(Q[t,S,D] = Spec_o(Q[t,S,D]) = Max_o(Q[t,S,D])$, where the last equality comes from theorem 2.2.

On the other hand if $P \in Spec_o(R[t,S,D])$ then obviously $P \cap \mathcal{S} = \varnothing$ and the inclusion $Spec_o(R[t,S,D]) \subseteq Max_o(R[t,S,D])$ is now and easy consequence of the fact that the (1.1) correspondence mentioned above preserves inclusion.■

Proposition 2.8. *Suppose that R satisfies d.c.c. on two-sided ideals. Then every ideal $P \in Spec_o(R[t,S,D])$ is closed.*

Proof. Let $0 \neq P \in Spec_\circ(R[t,S,D])$. First we will find a non-zero ideal I of R such that $I[P] \subset P$. For doing this, let us define $Q = \{h(t) \in T[t,S,D] \mid f_P(t)h(t) \in R[t,S,D]\}$ and $J = \{r \in R \mid f_P(t)r \in R[t,S,D]\}$. Clearly J is a non-zero ideal of R and $[P] = f_P(t)T[t,S,D] \cap R[t,S,D] = f_P(t)Q$. With the above notation we will prove that:

$$\begin{aligned} &\text{for any } m \geq 0 \text{ and } h(t) = \sum_{i=0}^{m} a_i t^i \in Q \\ &J^m h(t) \subset R[t,S,D] \quad . \end{aligned} \tag{$*$}$$

Let $h(t) = \sum_{i=0}^{m} a_i t^i \in Q$. Since $f_P(t)$ is monic and $f_P(t)h(t) \in R[t,S,D]$, $a_m \in R$. This establishes $(*)$ for $m = 0$.

Assume $m > 0$. By above, $a_m \in R$. Thus

$$f_P(t) J a_m t^m \subset f_P(t) J t^m \subset R[t,S,D] \quad .$$

Using this inclusion it is easy to see that $J(h(t) - a_m t^m) \subset Q$. Therefore, by inductive hypothesis, $J^{m-1}J(h(t) - a_m t^m) \subset R[t,S,D]$ and the statement $(*)$ follows.

R is a prime ring with d.c.c. on two-sided ideals, thus $J^m = J^{m+1} = \bar{J} \neq 0$ for some $m > 0$. Therefore, by $(*)$, $\bar{J}Q \subset R[t,S,D]$. Define $I = \{x \in S^n(\bar{J}) | x f_P(t) \in P,\ n = deg\, f_P(t)\}$. Then, by primeness of R, the ideal I is non-zero. Since $I f_P(t) \subset P$ and $S^{-n}(I) \subset \bar{J}$ with $n = deg\, f_P(t)$, we have:

$$I^2[P] = I^2 f_P(t) Q \subset (I f_P(t))(S^{-n}(I)Q) \subset PR[t,S,D] \subset P \quad .$$

Let $A = R[t,S,D]I^2R[t,S,D]$. Then A is an ideal of $R[t,S,D]$ having non-zero intersection with R and, by the above, $A[P] \subset P$. Now primeness of P implies $[P] = P$, i.e. P is closed. This establishes the proposition. ■

Now we will investigate the case when S and D commute. For this we will use a subring $T_\circ$ of T consisting of all such elements $q \in T$ that there is a non-zero S,D-stable ideal I of R such that $Iq, qI \subset R$ (one can look at $T_\circ$ as a Martindale symmetric quotient ring of R constructed with respect to the filter of all non-zero S,D-stable ideals of R). It is easy to see that $S(T_\circ) = T_\circ$ and $D(T_\circ) \subset T_\circ$. Therefore we can consider the following Ore extensions: $R[t,S,D] \subset T_\circ[t,S,D] \subset T[t,S,D]$.

We will continue to denote by $M(t) \in T[t,S,D]$ a monic invariant polynomial of minimal non-zero degree. As we remarked earlier, such a polynomial exists of $Spec_\circ(R[t,S,D]) \neq \{0\}$.

Proposition 2.9. *Suppose that S and D commute and that either $deg\, M(t) > 1$ or $S(M(t)) = M(t)$. Then every ideal $P \in Spec_\circ(R[t, S, D])$ is closed.*

Proof. Let $0 \neq P \in Spec_\circ(R[t, S, D])$. S commutes with D, thus we can apply Lemmas 1.8 and 1.9 to the polynomial $f_P(t)$ getting $f_P(t) \in T_\circ[t, S, D]$ and $f_P(t)t = t f_P(t)$. Now, using the fact that $R[t, S, D] \subset T_\circ[t, S, D]$ and $f_P(t) \in T_\circ[t, S, D]$, one can easily check that both $f_P(t)T_\circ[t, S, D]$ and $f_P(t)T[t, S, D]$ have the same intersection with $R[t, S, D]$. Therefore in order to prove that P is closed, it is enough to show that $P = f_P(t)T_\circ[t, S, D] \cap R[t, S, D]$. We will do this in two steps. First we will establish the following:

$$\hat{P} = f_P(t)R[t, S, D] \cap R[t, S, D] \subset P \quad . \qquad (*)$$

Consider $P_\circ = \{h(t) \in R[t, S, D] \mid f_P(t)h(t) \in P\}$. Clearly both $\hat{P}$ and $P_\circ$ are non–zero right ideals of $R[t, S, D]$. Since $f_P(t)$ commutes with t and $f_P(t)$ normalizes R, $f_P(t)$ also normalizes $R[t, S, D]$. This implies that $\hat{P}$ and $P_\circ$ are ideals of $R[t, S, D]$. Notice that $\hat{P}P_\circ \subset P$ but $P_\circ$ is not contained in P, because $P_\circ \cap R \neq 0$. Now primeness of P yields the statement $(*)$.

Let $g(t) \in f_P(t)T_\circ[t, S, D] \cap R[t, S, D]$. Then $g(t) = f_P(t)h(t)$ for some $h(t) \in T_\circ[t, S, D]$ and, by definition of $T_\circ$, there is a non–zero S, D–stable ideal J of R such that $Jh(t) \subset R[t, S, D]$. Since $S(J) = J$ and $f_P(t)$ is invariant, we have $Jf_P(t) = f_P(t)J$. Therefore

$$\begin{aligned} J\, g(t) = J\, f_P(t)h(t) = f_P(t)Jh(t) \subset \\ \subset f_P(t)R[t, S, D] \cap R[t, S, D] = \hat{P} \quad . \end{aligned} \qquad (**)$$

S, D–stability of J yields also that $\bar{J} = J\, R[t, S, D] = R[t, S, D]J$ is an ideal of $R[t, S, D]$. Using this together with $(**)$ and $(*)$ we get

$$\begin{aligned} \bar{J}(R[t, S, D]g(t)R[t, S, D]) \subset R[t, S, D]Jg(t)R[t, S, D] \subset \\ \subset R[t, S, D]\hat{P}R[t, S, D] \subset \hat{P} \subset P \quad . \end{aligned}$$

Because $\bar{J} \cap R \neq 0$, J is not included in P and primeness of P implies $g(t) \in P$. Thus $f_P(t)T_\circ[t, S, D] \cap R[t, S, D] \subset P$. This gives the proof of the proposition. ■

The last three propositions together with Lemma 2.6 and the remarks preceding Proposition 2.7 give us immediately the following:

Theorem 2.10. *Suppose that one of the following conditions is satisfied*

a) *R is symmetrically closed (i.e. $T = R$)*
b) *R is nœtherian*
c) *R satisfies D.C.C. on two-sided ideals*
d) *S and D commute and either $M(t) > 1$ or $S(M(t)) = M(t)$.*

Then every $P \in Spec_{\circ}(R[t, S, D])$ is closed and for a non zero R-disjoint ideal P of $R[t, S, D]$ the following conditions are equivalent :

(i) $P \in Spec_{\circ}(R[t, S, D])$.

(ii) $P \in Max_{\circ}(R[t, S, D])$.

(iii) $P = f(t)T[t, S, D] \cap R[t, S, D]$ *where the polynomial* $f(t) \in T[t, S, D]$ *is as described in Theorem 2.2 (iii).* ∎

Let us make a few final comments :

1) If $D = 0$ we can choose $M(t) = t$ so that $S(M(t)) = M(t)$ and the above theorem (case d)) applies.

2) Similarly, if $S = id$ we obviously have $S(M(t)) = M(t)$ and condition d) of the above theorem is satisfied.

3) More generally if S is the inner automorphism I_c of R induced by an invertible element $c \in R$ or if $D = D_{b;S}$ for some b in R then standard changes of variables show that $R[t, S = I_c, D] \cong R[t', D']$ and $R[t, S, D_{b;S}] = R[t'', D'']$ and hence the above theorem still applies.

4) We expect the conclusions of theorem 2.10 above to be true when S and D commute but one case is missed : the case when $SD = DS$, $deg\, M(t) = 1$, $S(M(t)) \neq M(t)$ and neither S nor D is inner on R, we cannot find an example satisfying all these conditions.(Notice that Lemma 1.5 (ii) shows that in such an example both S and D are inner on T).

5) The results of this section suggest that the same decription of $Spec(_{\circ}(R[t, S, D])$ as in the above theorem should hold for arbitrary Ore extension $R[t, S, D]$. Notice that such description exists if and only if every $P \in Spec_{\circ}(R[t, S, D])$ is closed.

Acknowledgment

The authors are very grateful to the referee for his helpful comments and suggestions.

References

1. A.D. Bell, *When are all prime ideals in an Ore extension Goldie ?*, Comm. Algebra **13(8)** (1985), 1743–1762.
2. W. Chin, *Prime ideals in differential operator rings and crossed products of infinite groups*, J. Algebra 106 (1987), 78–104.
3. M. Ferrero, *Prime and principal closed ideals of polynomial rings*, J. of Algebra **134 (1)** (1990), 45–59.
4. E. Cisneros, M. Ferrero, M.I. Gonzalez, *Prime ideals of skew polynomial rings and shew Laurent polynomial rings*, preprint.

5. M. Ferrero, J. Matczuk, *Prime ideals in skew polynomial rings of derivation type*, Comm. Algebra **18** (3) (1990), 689–710.
6. K.R. Gooddearl, *Prime ideals in shew polynomial rings and quantized*, Weyl algebras (to appear).
7. R.S. Irving, *Prime ideals of Ore extensions over commutative rings*, J. Algebra **56** (1979), 315–342.
8. R.S. Irving, *Prime ideals of Ore extensions over commutative rings II*, J. Algebra **58** (1979), 399–423.
9. A. Leroy, J. Matczuk, *The extended centroid and X-inner automorphisms of Ore extensions*, (to appear in J. Algebra).
10. M. Lorenz, D.S. Passman, *Prime ideals in crossed products of finite groups*, Israel J. Math. **33** (1979), 89–132.
11. J.C. McConnell, J.C. Robson, *Noncommutative Noetherian Rings*, Wiley–Interscience, New York (1987).
12. D.S. Passman, *Prime ideals in enveloping rings*, Trans. Amer. Math. Soc. 302(2) (1987), 535–560.